

Correspondence between field generalized binomial states and coherent atomic states

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Abstract. We show that the N -photon generalized binomial states of electromagnetic field may be put in a bijective mapping with the coherent atomic states of N two-level atoms. We exploit this correspondence to simply obtain both known and new properties of the N -photon generalized binomial states. In particular, an over-complete basis of these binomial states and an orthonormal basis are obtained. Finally, the squeezing properties of generalized binomial state are analyzed.

1 Introduction

Binomial states (BSs) constitute an important class of states originally introduced for the quantum electromagnetic (e.m.) field [1]. They are defined as a finite linear superposition of field number states $|n\rangle$ weighted by a binomial counting probability distribution. BSs are characterized by a maximum number of excitations N and a probability of single excitation occurrence p . When they are also characterized by a mean phase ϕ [2], they are called generalized binomial states (GBSs) [1,3]. One of their peculiarities is that they are “intermediate” between the number state and the coherent state. Because of their interesting features, BSs have been subject of several studies aimed at determining their properties and possible applications [1,2,4,5,6,7,8,9]. Recently, interest has again arisen about GBSs because of the discovery that they can be exploited as reference states within schemes devoted to measuring the canonical phase of quantum electromagnetic fields [10,11]. Moreover, in the context of cavity quantum electrodynamics (CQED), GBSs of e.m. field appear to be generated quite naturally and then used for fundamental or applicative purposes. In fact, they have been shown to be efficiently produced, within the current experimental capabilities [12,13], by standard atom-cavity interactions exploiting two-level atoms crossing one at a time a high- Q cavity [14,15]. A coherent orthogonal superposition of two GBSs of e.m. field is also a good candidate to study the classical-quantum border, due to the fact that each component of the superposition presents a non-zero mean field [15].

However, most of the properties of BSs and GBSs have been previously obtained by algebraic procedures [1,2,3,4,5,16,17], that often make non intuitive both the determination and interpretation of these properties. In particular, the orthogonality between two GBSs, that plays a fundamental rule in the study of classical-quantum superpositions, has been reported only recently [3]. GBSs have been also shown to be eigenstates of an eigenvalue equation using the Holstein-Primakoff realization of the Lie algebra $SU(2)$ [16]. This has led to the observation that they can be viewed as special $SU(2)$ coherent states [16,17]. It thus appears of interest to find simple and physically intuitive methods enabling us to get known and new properties of GBSs. This paper addresses this issue with reference to the GBSs of e.m. field.

To this end, we begin by noting that, due to the presence of a maximum number of excitations (photons) N , GBSs of e.m. field recall collective states describing excitations in systems of

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N atoms. We wish to show that this analogy can be put on solid grounds establishing a bijective mapping between the binomial states of N photons and the well-known coherent atomic states (CASs) of N two-level (spin-like) atoms [18,19]. Therefore, our target is to prove that GBSs are the electromagnetic correspondent of CASs. This correspondence aims at bringing at light properties of the GBSs by exploiting analogous properties possessed by the CASs. In particular, on the basis of a convenient geometrical representation, we succeed in interpreting GBSs as pseudo-angular momentum states. In order to obtain more knowledge about non-classical characteristics of GBSs, we finally analyze their squeezing behavior.

The paper is organized as follows. In Section 2 we give the definition of N -photon generalized binomial state (NGBS) and some of its principal properties. In Section 3 we show the bijection between NGBS and CAS, providing then an appropriate angular momentum operator approach for them. In Section 4, using this correspondence, we construct both an over-complete basis of NGBSs and an orthonormal basis of e.m. field states analogous to the rotated Dicke states. In Section 5 we analyze the squeezing behavior of the NGBSs and in Section 6 we report our conclusive discussions.

2 Generalized binomial states of e.m. field

The N -photon generalized binomial state (NGBS) is defined as [1,2,14,15]

$$|N, p, \phi\rangle = \sum_{n=0}^N \left[\binom{N}{n} p^n (1-p)^{N-n} \right]^{1/2} e^{in\phi} |n\rangle, \quad (1)$$

it is normalized and characterized by the probability of single photon occurrence $0 \leq p \leq 1$ and the mean phase ϕ . Note that the NGBS reduces to the vacuum state $|0\rangle$ when $p = 0$ and to the number state $|N\rangle$ when $p = 1$. In the particular case where $\phi = 0$, the NGBS of Eq. (1) is simply named “binomial state” [1]. On the other hand, for $N \rightarrow \infty$ and $p \rightarrow 0$ in such a way that $Np = \text{cost} \equiv |\alpha|^2$, the NGBS becomes the Glauber coherent state $||\alpha|e^{i\phi}\rangle$. Thus, as well-known a NGBS interpolates between the number and the coherent state and presents in general non-zero expectation values of the fields.

A relevant property for discussing the quantum-classical border is that, for a given NGBS it is possible to find a NGBS orthogonal to the first one. This is intuitively not obvious and, for example, it is a property not shared by coherent states. Orthogonality of two NGBSs has been recently reported using algebraic methods [3] and it has been shown that the inner product of two different NGBSs, $|N, p, \phi\rangle$ and $|N, p', \phi'\rangle$, is given by

$$\langle N, p, \phi | N, p', \phi' \rangle = \sum_{n=0}^N \binom{N}{n} (pp')^{n/2} [(1-p)(1-p')]^{(N-n)/2} e^{in(\phi' - \phi)}, \quad (2)$$

it thus vanishes when $p' = 1 - p$ and $\phi' = \pi + \phi$, that is [3]

$$\langle N, p, \phi | N, 1 - p, \pi + \phi \rangle = 0. \quad (3)$$

In the following, we shall exploit another important aspect of the NGBSs, that is the fact that they are the electromagnetic correspondent of the well-known coherent atomic states (CASs) [18]. Such a correspondence shall permit us to easily translate the properties of the CASs to the NGBSs.

3 Correspondence between N -photon generalized binomial states and coherent atomic states

A coherent atomic state (CAS) is a particular collective state of N two-level atoms that can be cast in the form [18]

$$|\theta, \varphi\rangle = \sum_{n=0}^{2J} \left[\binom{2J}{n} \left(\cos^2 \frac{\theta}{2} \right)^n \left(\sin^2 \frac{\theta}{2} \right)^{2J-n} \right]^{1/2} e^{-in\varphi} |J, -J + n\rangle, \quad (4)$$

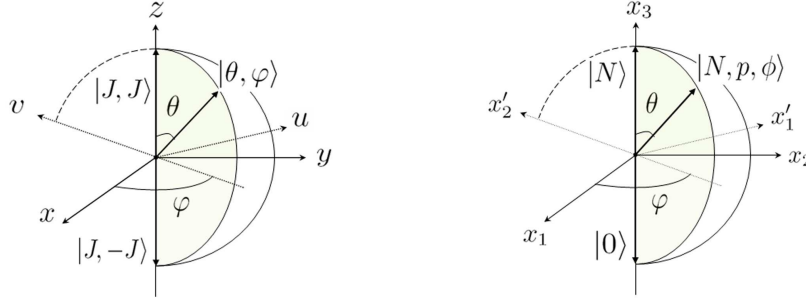


Fig. 1. Geometrical representations of the coherent atomic state (CAS) $|\theta, \varphi\rangle$ (left figure) and of the N -photon generalized binomial state (NGBS) $|N, p, \phi\rangle$ (right figure). The total angular momentum of the CAS points along the direction (θ, φ) . The probability of single photon occurrence p and the mean phase ϕ of the NGBS are linked to the polar angle θ and the azimuthal angle φ according to the relations $p \equiv \cos^2(\theta/2)$ and $\phi \equiv 2\pi - \varphi$.

where $|J, -J + n\rangle$ is the Dicke state of collective angular momentum J and excitation n with $0 \leq J \leq N/2$ if N is even or $1/2 \leq J \leq N/2$ if N is odd [19,21]. It is well-known that the CAS of Eq. (4) can be obtained by rotation of the highest excitation Dicke state $|J, J\rangle$ through an angle θ about an axis $\hat{\mathbf{u}} = (-\sin \varphi, \cos \varphi, 0)$ on the plane (x, y) , as shown in Appendix A and illustrated in the Bloch-sphere of Fig. 1. Thus, indicating by $R_{\theta, \varphi}$ the rotation operator $\exp\{-i\theta \mathbf{J} \cdot \hat{\mathbf{u}}\}$, we have $|\theta, \varphi\rangle \equiv R_{\theta, \varphi}|J, J\rangle$.

Glauber coherent states are usually considered the electromagnetic analogous of the coherent atomic states, because of common characteristics such as the formal generation by means of a displacement operator, the over-completeness property and the effective production by a classical source [19]. However, relevant differences occur between field coherent states and CASs. For example, for a fixed J , it is possible to find two orthogonal CASs $|\theta, \varphi\rangle$ and $|\pi - \theta, \pi + \varphi\rangle$ (see Appendix A), while two different coherent states are never orthogonal. Moreover, a CAS is represented by a finite superposition of Dicke states while a coherent state is represented by an infinite superposition of Fock states. In this section we show that the CASs have indeed an exact electromagnetic equivalent in the NGBSs.

By comparing the form of the NGBS given in Eq. (1) with that one of the CAS given in Eq. (4), it is readily seen that the two states assume exactly the same form if one associates the Dicke states $|J, -J + n\rangle$ to the number states $|n\rangle$ and sets the parametric correspondences

$$N \equiv 2J, \quad p \equiv \cos^2(\theta/2), \quad \phi \equiv 2\pi - \varphi \quad \Rightarrow \quad |N, p, \phi\rangle \leftrightarrow |\theta, \varphi\rangle. \quad (5)$$

This shows that the N -photon generalized binomial states can be put in a bijective mapping with the coherent atomic states. This permits us to easily transfer the properties of the CASs to the NGBSs. In particular, the NGBS may be geometrically represented in a pseudo Bloch-sphere as in Fig. 1 and formally obtained by a rotation of the highest number state $|N\rangle$.

This geometrical representation of the NGBS, together with the parametric associations of Eq. (5), allows to immediately obtain its limit properties, that is for $\theta = 0$ ($p = 1$) the NGBS becomes the number state $|N\rangle$ while for $\theta = \pi$ ($p = 0$) it becomes the vacuum (ground) state $|0\rangle$. Moreover, the orthogonality property of two NGBSs can be immediately determined from the corresponding one of the CASs, where two states are orthogonal if they point in opposite directions. $\langle \theta, \varphi | \pi - \theta, \pi + \varphi \rangle = 0$, and deduced from Fig. 1. In fact, the state $|N, p', \phi'\rangle$ “antiparallel” to $|N, p, \phi\rangle$ is described by the angles $\theta' = \pi - \theta$ and $\varphi' = \pi + \varphi$, which correspond respectively to the NGBS parameters $p' = 1 - p$ and $\phi' = \pi + \phi$, as seen from Eq. (5). This result gives $\langle N, p, \phi | N, 1 - p, \pi + \phi \rangle = 0$ as previously given by Eq. (3). In particular, one obtains that there is one and only one orthogonal state to a NGBS which is again a NGBS.

In order to follow our analogy, we now have to choose appropriate non-rotated angular momentum operators acting on the number (Fock) states analogous to the non-rotated atomic angular momentum operators J_z, J_{\pm} and then construct the rotated operators. The choice

naturally goes to the well-known Holstein-Primakoff realization of Lie algebra $SU(2)$ [22,16,20], which utilizes boson creation and annihilation operators a^\dagger, a and defines the pseudo-angular momentum operators acting on a NGBS in the following way:

$$J_3^{(N)} = a^\dagger a - N/2, \quad J_+^{(N)} = a^\dagger \sqrt{N - a^\dagger a}, \quad J_-^{(N)} = \sqrt{N - a^\dagger a} a. \quad (6)$$

The operators of Eq. (6) are the analogous of the non-rotated angular momentum operators given in Eq. (27) of Appendix A for the collective system of N two-level atoms and satisfy the same commutation rules

$$[J_+^{(N)}, J_-^{(N)}] = 2J_3, \quad [J_3^{(N)}, J_\pm^{(N)}] = \pm J_\pm^{(N)}, \quad [J_i^{(N)}, (J^{(N)})^2] = 0. \quad (7)$$

The action of these operators on the number state $|n\rangle$ is then

$$\begin{aligned} J_3^{(N)}|n\rangle &= (n - N/2)|n\rangle, \quad J_+^{(N)}|n\rangle = \sqrt{(N - n)(n + 1)}|n + 1\rangle, \\ J_-^{(N)}|n\rangle &= \sqrt{(N - n + 1)n}|n - 1\rangle, \end{aligned} \quad (8)$$

from which we readily obtain the eigenvalues equations

$$J_+^{(N)}|N\rangle = J_-^{(N)}|0\rangle = 0, \quad J_3^{(N)}|N\rangle = (N/2)|N\rangle, \quad J_3^{(N)}|0\rangle = (-N/2)|0\rangle. \quad (9)$$

In analogy with the case of CAS, the rotation operator $R_{\theta, \varphi}^{(N)} = \exp\{-i\theta \mathbf{J} \cdot \hat{\mathbf{x}}'_1\}$ that produces a rotation through an angle θ about an axis $\hat{\mathbf{x}}'_1 = (-\sin \varphi, \cos \varphi, 0)$ on the plane (x_1, x_2) , as illustrated in Fig. 1, results to be

$$R_{\theta, \varphi}^{(N)} = e^{-\eta J_+^{(N)} + \eta^* J_-^{(N)}}, \quad (10)$$

where $\eta = (\theta/2)e^{-i\varphi}$. The N -photon generalized binomial state $|N, p, \phi\rangle$ is then obtained by rotation of the number state $|N\rangle$ taking into account the parametric associations given by Eq. (5). In other words, we have

$$|N, p, \phi\rangle = R_{\theta, \varphi}^{(N)}|N\rangle = e^{-\eta J_+^{(N)} + \eta^* J_-^{(N)}}|N\rangle. \quad (11)$$

It is now very simple to find the rotated pseudo-angular momentum operators $J_3'^{(N)}, J_\pm'^{(N)}$ for the NGBSs. In fact, using the expressions of the rotated operators for the collective atomic system given in Eqs. (31) and (32) of Appendix A together with the parametric correspondences of Eq. (5), we have

$$\begin{aligned} J_3'^{(N)} &= (2p - 1)J_3^{(N)} + \sqrt{p(1 - p)}[e^{i\phi}J_+^{(N)} + e^{-i\phi}J_-^{(N)}], \\ J_+'^{(N)} &= [J_-'^{(N)}]^\dagger = e^{-i\phi}[pe^{i\phi}J_+^{(N)} - (1 - p)e^{-i\phi}J_-^{(N)} - 2\sqrt{p(1 - p)}J_3^{(N)}], \end{aligned} \quad (12)$$

from which we obtain the expected eigenvalues equations

$$\begin{aligned} J_3'^{(N)}|N, p, \phi\rangle &= (N/2)|N, p, \phi\rangle, \quad J_3'^{(N)}|N, 1 - p, \pi + \phi\rangle = -(N/2)|N, p, \phi\rangle, \\ J_+'^{(N)}|N, p, \phi\rangle &= J_-'^{(N)}|N, 1 - p, \pi + \phi\rangle = 0. \end{aligned} \quad (13)$$

In this sense, the NGBS $|N, p, \phi\rangle$ defined in Eq. (1) can be viewed as a pseudo-angular momentum state corresponding to the maximum value of the angular momentum $J = N/2$ along a direction (θ, φ) in a Bloch sphere, where the polar and azimuthal angles are fixed by the binomial state parameters as $\theta = 2 \arccos \sqrt{p}$ and $\varphi = 2\pi - \phi$. A similar form of the operator $J_3'^{(N)}$ given in Eq. (12) was previously obtained by using an algebraic ladder operator approach [16].

Analogously to the case of two CASs, two different NGBSs are linked by a rotation operator $T_{pp'\phi\phi'}$ in the pseudo Bloch-sphere as

$$|N, p', \phi'\rangle = T_{pp'\phi\phi'}|N, p, \phi\rangle. \quad (14)$$

The form of $T_{pp'\phi\phi'}$ is given by Eqs. (34) and (35) of Appendix A for the CASs case and using the parametric correspondences of Eq. (5).

4 N -photon generalized binomial states basis

In this section we shall focus our attention on an aspect that has not been previously discussed, that is the possibility to construct a basis of e.m. field states using N GBSs. The analogy with the CASs indicates both how to immediately obtain an over-complete basis of N GBSs and how to construct an orthonormal basis of pseudo-angular momentum states with two orthogonal N GBSs representing the ground and highest angular momentum state.

4.1 N GBSs over-complete basis

It is well-known that the CASs form an over-complete basis in the Hilbert space spanned by the $2J + 1$ Dicke states $|J, -J\rangle, |J, -J + 1\rangle, \dots, |J, J\rangle$, as described in detail in Appendix A [18,19]. As a consequence, the N GBSs can be also shown to form an over-complete basis in the space spanned by the $N + 1$ number states $|0\rangle, |1\rangle, \dots, |N\rangle$. In fact, by exploiting Eqs. (36) and (5) of Appendix A we get the completeness relation

$$(N + 1) \int \frac{d\Omega}{4\pi} |N, p, \phi\rangle \langle N, p, \phi| = 1, \quad (15)$$

where the infinitesimal solid angle $d\Omega$ can be written in terms of the N GBS parameters p, ϕ as

$$d\Omega = \sin \theta d\theta d\varphi = -2\sqrt{1-p} dp d\phi. \quad (16)$$

The expansion of an arbitrary e.m. field state $|\psi\rangle = \sum_n^N c_n |n\rangle$ in terms of N GBSs becomes

$$|\psi\rangle = (N + 1) \int \frac{d\Omega}{4\pi} \frac{A(\tau^*)}{[1 + |\tau|^2]^{N/2}} |N, p, \phi\rangle, \quad (17)$$

where $\tau = e^{i\phi} \sqrt{(1-p)/p}$ and the amplitude function $A(\tau^*)$ is given by

$$A(\tau^*) \equiv \sum_{n=0}^N c_n \binom{N}{n}^{1/2} (\tau^*)^n = [1 + |\tau|^2]^{N/2} \langle N, p, \phi | \psi \rangle. \quad (18)$$

These results indicate that the N GBSs may be used to construct a new representation for non-classical e.m. field states.

4.2 Complete orthonormal basis

The equivalence between CASs and N GBSs has permitted us to easily construct the rotated angular momentum operators $J_3'^{(N)}, J_{\pm}'^{(N)}$ given in Eq. (12). We have also seen that the two orthogonal N GBSs $|N, p, \phi\rangle, |N, 1-p, \pi + \phi\rangle$, related to direction (θ, φ) in Fig. 1, are the eigenstates of the operator $J_3'^{(N)}$ with eigenvalues $N/2$ and $-N/2$, respectively. Thus, they represent the highest and ground state of the orthonormal basis formed by all the eigenstates of the operator $J_3'^{(N)}$. It is then possible to obtain the expression of these e.m. field states, each orthogonal to the two orthogonal N GBSs, that represent the electromagnetic analogous of the rotated Dicke states.

This goal can be reached by repeatedly applying the rotated raising operator $J_+'^{(N)}$ of Eq. (12) on the ground state $|N, 1-p, \pi + \phi\rangle$. In fact, because of the commutation rules given in Eq. (7), we also have that the operator J^2 commutes with $J_+'^{(N)} = 0$, so that the action of the rotated raising operator is inside the subspace $J = N/2$. We then obtain the orthonormal

basis of e.m. field states $\mathcal{B} = \{|\Delta_{p,\phi}(N/2, m - N/2)\rangle, m = 0, 1, \dots, N\}$, which we call “Delta” states, defined as

$$\begin{aligned} |\Delta_{p,\phi}(N/2, m - N/2)\rangle &\equiv \binom{N}{m}^{-1/2} \frac{[J_+^{(N)}]^m}{m!} |N, 1 - p, \pi + \phi\rangle \\ &= \frac{J_+^{(N)}}{\sqrt{m(N - m + 1)}} |\Delta_{p,\phi}(N/2, m - 1 - N/2)\rangle \end{aligned} \quad (19)$$

where we have set $|\Delta_{p,\phi}(N/2, -N/2)\rangle \equiv |N, 1 - p, \pi + \phi\rangle$ and $|\Delta_{p,\phi}(N/2, N/2)\rangle \equiv |N, p, \phi\rangle$.

As an example, let us consider the case $N = 2$. The three Delta states forming the basis are the two orthogonal 2GBSs $|2, p, \phi\rangle, |2, 1 - p, \pi + \phi\rangle$, given by Eq. (1), and the intermediate state $|\Delta_{p,\phi}(1, 0)\rangle$ obtained by Eq. (19) and having the form

$$|\Delta_{p,\phi}(1, 0)\rangle = \sqrt{2p(1 - p)}|0\rangle + (2p - 1)e^{i\phi}|1\rangle - \sqrt{2p(1 - p)}e^{i2\phi}|2\rangle. \quad (20)$$

We point out that the previous Delta state arises in the process of generation of entangled 2GBSs in two separate cavities and in principle it can be conditionally obtained during that process [23].

Therefore, the analogy between coherent atomic states and N -photon generalized binomial states has allowed us to introduce a new class of e.m. field states that we have named “Delta” states which are the analogous of rotated Dicke states.

5 Squeezing of N -photon generalized binomial states

In this section we shall analyze the squeezing behavior of the N -photon generalized binomial states, in order to characterized some of their nonclassical properties.

The squeezing of a single-mode e.m. field state are usually examined taking into account the two field quadratures a_X, a_P defined as [19]

$$a_X = a + a^\dagger, \quad a_P = \frac{a - a^\dagger}{i}, \quad (21)$$

where a, a^\dagger are respectively the annihilation and creation operators of the field mode. The two quadratures a_X, a_P behave like dimensionless canonical conjugates, satisfying the following commutation and uncertainty relations for any field state:

$$[a_X, a_P] = 2i, \quad \langle \Delta a_X^2 \rangle \langle \Delta a_P^2 \rangle \geq 1, \quad (22)$$

where the dispersion $\langle \Delta a_K^2 \rangle$ ($K = X, P$) is given by $\langle \Delta a_K^2 \rangle = \langle a_K^2 \rangle - \langle a_K \rangle^2$. A squeezed state is defined as a state for which either a_X or a_P has a dispersion below unity with a corresponding increase in the dispersion of the other quadrature [19]. In order to investigate the squeezing behavior of NGBSs, that are not states of minimum uncertainty for the quadratures, it is convenient to introduce the squeezing indexes S_X, S_P as

$$S_K = 1 - \langle \Delta a_K^2 \rangle, \quad (K = X, P) \quad (23)$$

so that squeezing occurs when $S_K > 0$.

Applying the previous considerations to the NGBS $|N, p, \phi\rangle$ defined in Eq. (1) we obtain

$$\begin{aligned} S_X &= -2Np - A(N, p) \cos 2\phi + B^2(N, p) \cos^2 \phi, \\ S_P &= -2Np + A(N, p) \cos 2\phi + B^2(N, p) \sin^2 \phi, \end{aligned} \quad (24)$$

where we have set

$$\begin{aligned} A(N, p) &= 2\sqrt{N(N - 1)}p(1 - p) \sum_{n=0}^{N-2} \left[\binom{N}{n} \binom{N - 2}{n} \right]^{1/2} p^n (1 - p)^{N-2-n}, \\ B(N, p) &= 2\sqrt{Np(1 - p)} \sum_{n=0}^{N-1} \left[\binom{N}{n} \binom{N - 1}{n} \right]^{1/2} p^n (1 - p)^{N-1-n}. \end{aligned} \quad (25)$$

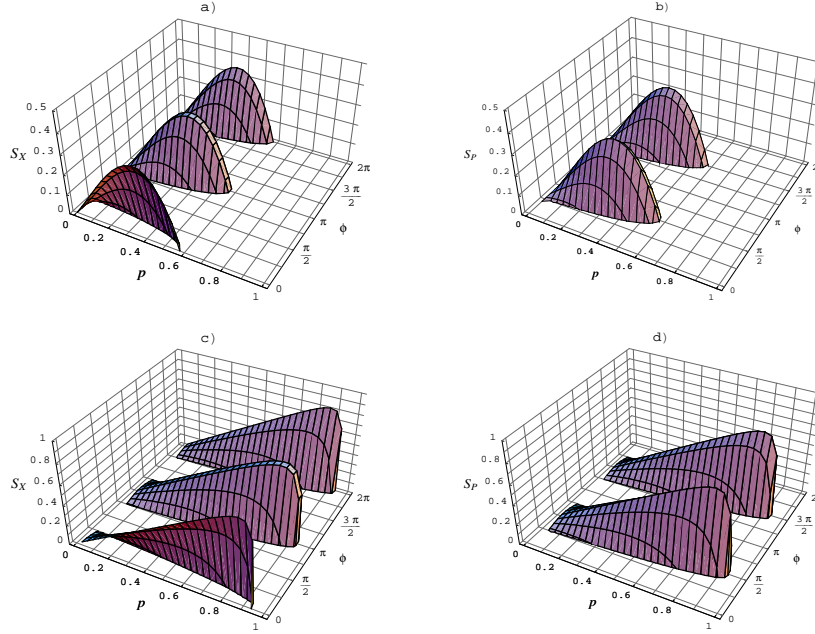


Fig. 2. Squeezing indexes S_X [plots (a) and (c)] and S_P [plots (b) and (d)] as a function of the probability of single photon occurrence p and the mean phase ϕ , for $N = 2$ [plots (a) and (b)] and $N = 100$ [plots (c) and (d)]. Squeezing of the quadratures a_X or a_P respectively occurs when $S_X > 0$ or $S_P > 0$.

The squeezing behavior is analyzed by fixing the value of the maximum number of photons N and varying the value of the parameters p, ϕ . We find that squeezing occurs, that is $S_X > 0$ or $S_P > 0$, for some values of p, ϕ at any value of $N \geq 1$. We also find that the larger is N , the more squeezed is the quadrature a_X or a_P . We represent this behavior in Fig. 2, where we plot the squeezing indexes S_X and S_P as functions of the probability of single photon occurrence p and the mean phase ϕ , for the two values $N = 2$ and $N = 100$. We observe that, while the range of values of ϕ where squeezing occurs remain unchanged for different N , the squeezing range of p increases with N . We finally note that, when the quadrature a_X is squeezed, the quadrature a_P is not squeezed and viceversa.

Therefore, we may conclude that the N -photon generalized binomial states are particular squeezed states of the e.m. field.

6 Conclusion

In this paper we have shown that the N -photon generalized binomial states (NGBSs) of e.m. field are the electromagnetic correspondent of the coherent atomic states (CASs) describing a set of N two-level atoms. We have shown that this equivalence simply arises from an association of the parameters characterizing the two kind of states, that is NGBSs and CASs have the same structure (Sec. 3). This correspondence has then permitted us to immediately apply the known results for the CASs to the NGBSs.

Using this analogy, we have described the NGBSs as pseudo-angular momentum states, representable in a Bloch sphere and obtainable by a rotation in the space of the number state $|N\rangle$. We have also found the appropriate rotated angular momentum operators for NGBSs by using the Holstein-Primakoff operators as non-rotated angular momentum operators acting on the number states. This has permitted to immediately obtain an eigenvalue equation for the NGBSs (Sec. 3). Note that an eigenvalue equation for binomial states was previously obtained by means of a ladder operator approach [16].

Exploiting the analogy with the CAS, we have then constructed an over-complete basis of NGBSs in the Hilbert space spanned by the first $N + 1$ number states (Sec. 4), showing that an arbitrary single-mode field state with a maximum photon number N can be expressed in terms of this basis.

The bijective mapping between NGBS and CAS has allowed to construct an orthonormal basis of e.m. field states that are eigenstates of the rotated pseudo-angular momentum operator. In this basis, two orthogonal NGBSs constitute the ground and highest angular momentum state. These states, named “Delta”, are a new class of e.m. field states that correspond to rotated Dicke states of a N two-level atoms system (Sec. 4). Moreover, they may be generated by standard atom-cavity interactions in the CQED framework [14].

Finally, we have analyzed the squeezing properties of the NGBSs (Sec. 5). We have found that the NGBSs are particular squeezed states of the e.m. field. In fact, squeezing of the field quadratures occurs at some values of the NGBS parameters for a given value of the maximum number of photons $N \geq 1$.

In conclusion, the perfect correspondence with CASs has allowed us to obtain immediately the principal properties and also new characteristics of NGBSs. The NGBSs are thus the “true” electromagnetic analogous of the CASs. The usual correspondence between Glauber coherent states and coherent atomic states may be then viewed as a consequence of the fact that Glauber coherent states are somehow a limit of the NGBSs (see Sec. 2). The analogy with the CASs appears to be the underlying reason why NGBSs are naturally generated by interactions between two-level atoms and a quantum e.m. field initially in its vacuum (ground) state, as shown by the feasible efficient generation schemes recently proposed in the context of CQED [14,15], based on two-level atoms interacting one at a time with a high- Q cavity. These generation methods have been used to reach a 2GBS. However, the analogy with CASs suggest that the generation of NGBSs with $N > 2$ is also possible in the CQED framework and this will be treated elsewhere. Finally, in view of the correspondence between NGBS and CAS, it would be also of interest to investigate if a NGBS can be produced in a single-shot run by the interaction of N two-level atoms in a collective coherent atomic state with a quantum e.m. field.

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A Dicke states and Coherent atomic states

In this Appendix we review the definition of coherent atomic state (CAS) and give some of its properties.

Let us consider N identical two-level (spin-like) atoms described by usual spin operators $s_i^j = \sigma_i^j/2$ ($i = x, y, z; j = 1, 2, \dots, N$), where σ_i^j is the i -th Pauli matrix relating to the j -th atom, or by the atomic lowering and raising operators $\sigma_-^j = |g_j\rangle\langle e_j|$, $\sigma_+^j = |e_j\rangle\langle g_j|$, with g and e representing respectively the ground and excited state of the two-level atom. The collective system of these N atoms has the corresponding Hilbert space spanned by the set of 2^N product states

$$|\Phi_{k_1 k_2 \dots k_N}\rangle = \prod_{j=1}^N |k_j\rangle. \quad (k_j = g_j, e_j) \quad (26)$$

It is then convenient to introduce collective spin operators as

$$J_i \equiv \sum_{j=1}^N s_i^j, \quad J_{\pm} \equiv \sum_{j=1}^N \sigma_{\pm}^j = J_x \pm iJ_y, \quad J^2 = J_x^2 + J_y^2 + J_z^2, \quad (27)$$

satisfying the commutation relations

$$[J_+, J_-] = 2J_z, \quad [J_z, J_{\pm}] = \pm J_{\pm}, \quad [J_i, J^2] = 0. \quad (28)$$

Note that the J_+ , J_- operators raise and lower the excitation of the collective atomic system by unity in a manner such that this excitation is distributed over all atoms. At this point, one constructs the Dicke states $|J, M\rangle$ [21], which are simply the usual angular momentum states, defined as the eigenstates of J_z , J^2 with eigenvalues respectively M , $J(J+1)$ and given by

$$|J, M = -J + n\rangle \equiv \frac{1}{n!} \binom{2J}{n}^{-1/2} J_+^n |J, -J\rangle, \quad (n = 0, 1, \dots, 2J) \quad (29)$$

where the ground state $|J, -J\rangle$ and the highest excitation state $|J, J\rangle$ are defined by $J_-|J, -J\rangle = J_+|J, J\rangle = 0$, with $\min|M| \leq J \leq N/2$. When $J = N/2$ the number n of Eq. (29) can be viewed as the atomic excitation, so that the Dicke states represent states of definite atomic excitation in which the excitation is however distributed among the different atoms. In this sense, the Dicke states of the N -atom system can be considered as analogous to the Fock states of the e.m. field, the atomic excitation being correspondent to the photon excitation [19].

Let us now consider the rotation operator $R_{\theta, \varphi} = \exp\{-i\theta \mathbf{J} \cdot \hat{\mathbf{u}}\}$ producing a rotation through an angle θ about an axis $\hat{\mathbf{u}} = (-\sin \varphi, \cos \varphi, 0)$ on the plane (x, y) , as illustrated in Fig. 1. It is possible to show that this operator can be factorized using the disentangling theorem for angular momentum operators as [18]

$$R_{\theta, \varphi} = e^{-(\xi J_+ - \xi^* J_-)} = e^{\tau^* J_-} e^{-\ln(1+|\tau|^2) J_z} e^{-\tau J_+}, \quad (30)$$

where $\xi = (\theta/2)e^{-i\varphi}$ and $\tau = \tan(\theta/2)e^{-i\varphi}$. In general, the action of this rotation operator on an arbitrary atomic state $|c\rangle$ or an atomic operator Q reads like $|c'\rangle = R_{\theta, \varphi}|c\rangle$ and $Q' = R_{\theta, \varphi} Q R_{\theta, \varphi}^{-1}$. The coherent atomic state $|\theta, \varphi\rangle$ given in Eq. (4) is then obtained by rotation of the highest excitation Dicke state $|J, J\rangle$, that is $|\theta, \varphi\rangle \equiv R_{\theta, \varphi}|J, J\rangle$. This CAS can be equivalently generated by rotation of the ground state $|J, -J\rangle$ through an angle $\theta' = \pi - \theta$ about an axis $\hat{\mathbf{u}}'$ antiparallel to the previous one $\hat{\mathbf{u}}$.

Note that for $\theta = 0$ the CAS reduces to the Dicke state $|J, J\rangle$ while for $\theta = \pi$ it reduces to the lowest excitation (ground) Dicke state $|J, -J\rangle$. The coherent atomic state $|\theta, \varphi\rangle$ is thus eigenstate of the rotated angular momentum operator J'_z with eigenvalue J , that is $J'_z|\theta, \varphi\rangle = J|\theta, \varphi\rangle$, where

$$J'_z = J_z \cos \theta + \sin \theta (J_+ e^{-i\varphi} + J_- e^{i\varphi})/2. \quad (31)$$

Of course, the eigenvalue equation $J^2|\theta, \varphi\rangle = J(J+1)|\theta, \varphi\rangle$ holds, as well. At the same way, we can find the rotated raising and lowering operators J'_\pm which have the form

$$J'_+ = (J'_-)^{\dagger} = e^{i\varphi} [J_+ e^{-i\varphi} \cos^2(\theta/2) - J_- e^{i\varphi} \sin^2(\theta/2) - J_z \sin \theta]. \quad (32)$$

From Eqs. (32) and (4) one then obtains the eigenvalue equation $J'_+|\theta, \varphi\rangle = 0$, as expected.

It is also useful to have the global operator that links two different CASs, that is $|\theta', \varphi'\rangle = T_{\theta\theta', \varphi\varphi'}|\theta, \varphi\rangle$. This operator can be found by observing that, from the definition of CAS, we have

$$|\theta', \varphi'\rangle = R_{\theta', \varphi'}|J, J\rangle = R_{\theta', \varphi'} R_{\theta, \varphi}^{-1}|\theta, \varphi\rangle, \quad (33)$$

from which one obtains [18]

$$T_{\theta\theta', \varphi\varphi'} \equiv R_{\theta', \varphi'} R_{\theta, \varphi}^{-1} = \exp \left\{ i \frac{\theta\theta'}{4} \sin(\varphi - \varphi') \right\} R_{\Theta, \Phi}, \quad (34)$$

where the angles Θ, Φ are given by

$$\Theta = [\theta^2 + \theta'^2 - 2\theta\theta' \cos(\varphi - \varphi')]^{1/2}, \quad \tan \Phi = \frac{\theta' \sin \varphi' - \theta \sin \varphi}{\theta' \cos \varphi' - \theta \cos \varphi}. \quad (35)$$

The coherent atomic states also form an over-complete set and they can be exploited as basis. In fact, using the completeness of Dicke states $\sum_n^{2J} |J, -J+n\rangle \langle J, -J+n| = 1$ it is found that [18]

$$(2J+1) \int \frac{d\Omega}{4\pi} |\theta, \varphi\rangle \langle \theta, \varphi| = 1, \quad (36)$$

and the expansion of an arbitrary atomic state $|c\rangle = \sum_n^{2J} c_n |J, -J+n\rangle$ in terms of CASs reads like

$$|c\rangle = (2J+1) \int \frac{d\Omega}{4\pi} \frac{f(\tau^*)}{[1+|\tau|^2]^J} |\theta, \varphi\rangle, \quad (37)$$

where the amplitude function $f(\tau^*)$ is given by

$$f(\tau^*) \equiv \sum_n^{2J} c_n \binom{2J}{n}^{1/2} (\tau^*)^n = [1+|\tau|^2]^J \langle \theta, \varphi | c \rangle. \quad \left(\tau = \tan \frac{\theta}{2} e^{-i\varphi} \right) \quad (38)$$

Two different CASs are not orthogonal in general, that is $\langle \theta, \varphi | \theta', \varphi' \rangle \neq 0$. However, from the geometrical representation of Fig. 1 it is readily seen that the CAS $|\theta, \varphi\rangle$ of Eq. (4) admits an “antiparallel” orthogonal CAS given by $|\pi - \theta, \pi + \varphi\rangle$, that is $\langle \theta, \varphi | \pi - \theta, \pi + \varphi \rangle = 0$. In particular, the scalar product of two CASs is given by [18,19]

$$|\langle \theta, \varphi | \theta', \varphi' \rangle|^2 = [\cos(\Theta/2)]^{4J}, \quad (39)$$

where Θ is the angle between the two CASs.

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